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Unified model based on $U(1)$ duality symmetry of polarization and magnetization

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The exact similarity observed between the external field lines of an electric dipole and those due to a magnetic dipole can be explained by a $U(1)$ duality symmetry that internally couples the polarization and magnetization sources. This symmetry provides an approach of performing magnetic calculations by mapping them to electric calculations and vice versa. A classical model based on this symmetry, which describes and unifies the external electric and magnetic fields of material sources, is presented in this paper. This model predicts that an external magnetostatic dipole field is equally well described as originating from a particle possessing a magnetic dipole moment or as due to a particle having both electric and magnetic dipole moments. [S1063-651X(98)51112-9]

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Duality is an old idea that is unexpectedly changing the way of looking at the physical world [1,2]. By duality symmetry two apparently dissimilar theories become dual; that is, a single theory with two equivalent descriptions [3–5]. This enables a considerable mathematical simplification since questions in one of the theories might be elucidated by making calculations with the other and vice versa [1,5]. This has lead to the belief that duality might be the key for unifying all forces [6]. The original idea of electromagnetic duality arose with Maxwell's equations in absence of sources, which can concisely be written (in Gaussian units) as

$$\nabla\cdot(\mathbf{E}+i\mathbf{B})=0,\quad \nabla\times(\mathbf{E}+i\mathbf{B})-\frac{i}{c}\frac{\partial}{\partial t}(\mathbf{E}+i\mathbf{B})=0. \quad (1)$$

These equations are invariant under the duality transformations $(\mathbf{E}'+i\mathbf{B}')=e^{i\theta}(\mathbf{E}+i\mathbf{B})$, where θ is a real parameter. To preserve this $U(1)$ symmetry in the presence of electric sources $(\rho_e,\mathbf{J}_e)$, it is necessary to include magnetic sources $(\rho_m,\mathbf{J}_m)$ as well. Thus, the generalized Maxwell equations

$$\nabla\cdot(\mathbf{E}+i\mathbf{B})=4\pi(\rho_e+i\rho_m),$$
$$\nabla\times(\mathbf{E}+i\mathbf{B})-\frac{i}{c}\frac{\partial}{\partial t}(\mathbf{E}+i\mathbf{B})=\frac{4\pi i}{c}(\mathbf{J}_e+i\mathbf{J}_m), \quad (2)$$

are invariant under duality transformations of the fields $\mathbf{E}$ and $\mathbf{B}$ provided the sources are transformed in the same way: $(\mathbf{J}_e'+i\mathbf{J}_m')=e^{i\theta}(\mathbf{J}_e+i\mathbf{J}_m)$ and $(\rho_e'+i\rho_m')=\mathrm{e}^{i\theta}(\rho_e+i\rho_m)$. These reduce to $(q_e'+iq_m')=e^{i\theta}(q_e+iq_m)$ for matter composed of particles with electric (q_e) and magnetic (q_m) charges. In nature, however, electromagnetic duality appears to be spoiled by the fact that one observes electric charges but not magnetic ones [6–8]. On this point it has been argued that duality is essentially quantum in nature with no classical counterpart [7], that it makes more sense with supersymmetry [8], and that it is much more natural in string theory [2]. In general terms, duality in quantum theories enables the exchange of electric and magnetic fields, $\mathbf{E}\leftrightarrow\mathbf{B}$, by exchanging magnetic and electric charges, $q_e\leftrightarrow q_m$, and exchanging the coupling constants $\alpha\leftrightarrow 1/\alpha$ associated to q_e and q_m. Since α is small (in usual quantum theories) then $1/\alpha$ is large. Hence, duality relates a weakly coupled theory to a strongly coupled theory. This idea has impacted the actual quantum theories [5].

The above field-theory views have overlooked that duality is factually present in the classical electrodynamics of dipoles. The fact that the external-field lines of an electric dipole cannot be distinguished from those due to a magnetic dipole is a clear manifestation of this symmetry, which is observed in the static regime and continues being exact even when the static dipoles have time-varying moments and, moreover, when they are in arbitrary motion [9–11]. This is an exact duality symmetry provided by nature. The pertinent question arises: What is the theory associated with the dipole symmetry? The usual equations describing the classical electrodynamics of dipoles are, of course, Maxwell's equations with polarization and magnetization sources. Nevertheless, these equations are not duality invariant.

A classical electromagnetic model based on a $U(1)$ duality symmetry that couples polarization and magnetization sources is constructed in this Rapid Communication. This model describes and unifies the external electric and magnetic fields of material sources. In this model the electrostatics of polarization sources is internally connected with the magnetostatics of magnetization sources in such a way that one can perform magnetic calculations by mapping them to electric calculations and vice versa. The model predicts that an external magnetostatic dipole field is equally well described as originated by a particle possessing a magnetic dipole moment or as due to a particle having both electric and magnetic dipole moments.

Although Maxwell's equations with polarization ($\mathbf{P}$) and magnetization ($\mathbf{M}$) sources

$$\nabla\cdot\mathbf{E}=-4\pi\nabla\cdot\mathbf{P}, \quad (3a)$$

$$\nabla\cdot\mathbf{B}=0, \quad (3b)$$

$$\nabla\times\mathbf{E}+1/c\ (\partial\mathbf{B}/\partial t)=0, \quad (3c)$$

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{4\pi}{c}\left(c\nabla\times\mathbf{M}+\frac{\partial\mathbf{P}}{\partial t}\right) \quad (3d)$$

*Electronic address: heras@fenix.ifisicacu.unam.mx describe the electrodynamics of dipoles, they are not duality invariant. The complex vector $(\mathbf{P}+i\mathbf{M})$ cannot consistently be introduced, and so Eqs. (3) cannot be written as Eqs. (2); the vectors $\mathbf{P}$ and $\mathbf{M}$ are the sources of the fields $\mathbf{E}$ and $\mathbf{B}$. Hence $\mathbf{P}$ and $\mathbf{M}$ are defined independently of the proper fields $\mathbf{E}$ and $\mathbf{B}$ as electric and magnetic dipole moment densities. Consider now the nontrivial question, Why are Eqs. (3) not duality invariant even when they describe the electrodynamics of dipoles? The answer can be found by solving these equations. Using the generalized Helmholtz's theorem [12,13] the solution of Eqs. (3) is

$$\mathbf{E}=\int\int\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{P})-\mathbf{P}}{R^3}+\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{P}})-\dot{\mathbf{P}}+\mathbf{n}\times\dot{\mathbf{M}}}{R^2c}+\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{P}}+\ddot{\mathbf{M}})}{Rc^2}\right)\delta(u)d^3x'dt'-\frac{4\pi}{3}\mathbf{P}, \tag{4a}$$

$$\mathbf{B}=\int\int\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{M})-\mathbf{M}}{R^3}+\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{M}})-\dot{\mathbf{M}}-\mathbf{n}\times\dot{\mathbf{P}}}{R^2c}+\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{M}}-\ddot{\mathbf{P}})}{Rc^2}\right)\delta(u)d^3x'dt'+\frac{8\pi}{3}\mathbf{M}, \tag{4b}$$

where the time integration is from $-\infty$ to $+\infty$ and the spatial integration is over all space; $\delta(u=t'+R/c-t)$ provides the retardation; $\mathbf{n}=\mathbf{R}/R$ with $\mathbf{R}=(\mathbf{x}-\mathbf{x}')$ and $R=|\mathbf{x}-\mathbf{x}'|$; and overdots mean differentiation with respect to t'. The integral terms in Eqs. (4) represent the fields outside the material sources and the contact terms $[-(4\pi/3)\mathbf{P}$ and $+(8\pi/3)\mathbf{M}]$ represent the fields inside the sources. Evidently, the integral terms are invariant under duality transformations, whereas the contact terms are not. Since the integral terms give rise the external fields of dipoles, Maxwell's theory is consistent with the exact similarity of the external-field lines of dipoles. The problem with the theory is that the asymmetry of the associated contact terms breaks the duality symmetry of the full solutions. The presence of the contact terms is the reason why Maxwell's equations are not duality invariant. The origin of the contact terms is intimately related to the structure of Eqs. (3); notice that the solution of the duality-invariant Eqs. (2) does not involve contact terms [12,14]. Hence, the simplest way to construct a model based on the $U(1)$ duality symmetry of dipoles consists of modifying Eqs. (3) in such a way that the solution of the modified equations is given by the integral terms of Eqs. (4). Consider the general duality-invariant Maxwell-like equations

$$\nabla\cdot\mathbb{E}=k_1\nabla\cdot\mathbf{P}, \tag{5a}$$

$$\nabla\cdot\mathbb{B}=k_1\nabla\cdot\mathbf{M}, \tag{5b}$$

$$\nabla\times\mathbb{E}+1/c\,(\partial\mathbb{B}/\partial t)=k_2\nabla\times\mathbf{P}-k_3(\partial\mathbf{M}/\partial t), \tag{5c}$$

$$\nabla\times\mathbb{B}-1/c\,(\partial\mathbb{E}/\partial t)=k_2\nabla\times\mathbf{M}+k_3(\partial\mathbf{P}/\partial t), \tag{5d}$$

where $\mathbb{E}$ and $\mathbb{B}$ are the electric and magnetic fields and k_1, k_2, and k_3 are parameters to determine. The integration of these equations gives the following expression for the field $\mathbb{E}$:

$$\mathbb{E}=\frac{k_2-k_1}{4\pi}\int\int\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{P})-\mathbf{P}}{R^3}+\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{P}})-\dot{\mathbf{P}}}{R^2c}+\frac{\mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{P}})}{Rc^2}\right)\times\delta(u)d^3x'dt'+\frac{k_2+ck_3}{4\pi}\int\int\left(\frac{\mathbf{n}\times\dot{\mathbf{M}}}{R^2c}+\frac{\mathbf{n}\times\ddot{\mathbf{M}}}{Rc^2}-\frac{\ddot{\mathbf{P}}}{Rc^2}\right)\delta(u)d^3x'dt'+\left(\frac{k_1+2k_2}{3}\right)\mathbf{P}. \tag{6}$$

The condition to eliminate the contact term is $k_1+2k_2=0$ and the conditions to reproduce the integral term of Eq. (4a) are $k_2-k_1=4\pi$ and $k_2+ck_3=4\pi$. The solution of these algebraic equations is given by $k_1=-8\pi/3$, $k_2=4\pi/3$, and $k_3=8\pi/3c$. With these values of the parameters the field equations take the form

$$\nabla\cdot\mathbb{E}=-\frac{8\pi}{3}\nabla\cdot\mathbf{P}, \tag{7a}$$

$$\nabla\cdot\mathbb{B}=-\frac{8\pi}{3}\nabla\cdot\mathbf{M}, \tag{7b}$$

$$\nabla\times\mathbb{E}+\frac{1}{c}\frac{\partial\mathbb{B}}{\partial t}=\frac{4\pi}{3c}\left(c\nabla\times\mathbf{P}-2\frac{\partial\mathbf{M}}{\partial t}\right), \tag{7c}$$

$$\nabla\times\mathbb{B}-\frac{1}{c}\frac{\partial\mathbb{E}}{\partial t}=\frac{4\pi}{3c}\left(c\nabla\times\mathbf{M}+2\frac{\partial\mathbf{P}}{\partial t}\right), \tag{7d}$$

and the solution for fields $\mathbb{E}$ and $\mathbb{B}$ takes the desired form

$$\mathbb{E}=\int\int\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{P})-\mathbf{P}}{R^3}+\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{P}})-\dot{\mathbf{P}}+\mathbf{n}\times\dot{\mathbf{M}}}{R^2c}+\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{P}}+\ddot{\mathbf{M}})}{Rc^2}\right)\delta(u)d^3x'dt', \tag{8a}$$

$$\mathbb{B}=\int\int\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{M})-\mathbf{M}}{R^3}+\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{M}})-\dot{\mathbf{M}}-\mathbf{n}\times\dot{\mathbf{P}}}{R^2c}+\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{M}}-\ddot{\mathbf{P}})}{Rc^2}\right)\delta(u)d^3x'dt'. \tag{8b}$$

These equations determine completely the fields $\mathbb{E}$ and $\mathbb{B}$ [15]. It is now evident that Maxwell's fields $\mathbf{E}$ and $\mathbf{B}$ and fields $\mathbb{E}$ and $\mathbb{B}$ of the proposed model are related by

$$\mathbf{E}=\mathbb{E}-(4\pi/3)\,\mathbf{P}$$

and

$$\mathbf{B}=\mathbb{B}+(8\pi/3)\,\mathbf{M}. \tag{9}$$

In this way fields $\mathbb{E}$ and $\mathbb{B}$ describe the electric and magnetic fields outside the sources while the contact terms give the fields inside the sources. Evidently, outside the sources fields $\mathbf{E}$ and $\mathbf{B}$ coincide with fields $\mathbb{E}$ and $\mathbb{B}$. Equations (7) can concisely be written as

$$\nabla\cdot(\mathbb{E}+i\mathbb{B})=-(8\pi/3)\,\nabla\cdot(\mathbf{P}+i\mathbf{M}), \tag{10a}$$

$$\nabla\times(\mathbb{E}+i\mathbb{B})-i/c\,(\partial/\partial t)(\mathbb{E}+i\mathbb{B})=\frac{4\pi}{3c}\left(c\nabla\times(\mathbf{P}+i\mathbf{M})+2i\frac{\partial}{\partial t}(\mathbf{P}+i\mathbf{M})\right), \tag{10b}$$

which are invariant under the duality transformations: $(\mathbb{E}'+i\mathbb{B}')=e^{i\theta}(\mathbb{E}+i\mathbb{B})$ provided the material sources $\mathbf{P}$ and $\mathbf{M}$ transform as $(\mathbf{P}'+i\mathbf{M}')=e^{i\theta}(\mathbf{P}+i\mathbf{M})$, which implies $(\mathbf{d}'+i\boldsymbol{\mu}')=e^{i\theta}(\mathbf{d}+i\boldsymbol{\mu})$, where $\mathbf{d}$ and $\boldsymbol{\mu}$ denote the electric and magnetic dipole moments. Since Eqs. (7) and their solution in Eqs. (8) *are duality invariant*, then the observed duality symmetry of dipoles is *consistently* explained by the electromagnetic model presented here. To see that this model is already a unified theory consider the static limit of Eqs. (7),

$$\boldsymbol{\nabla}\cdot\mathbb{E}=-\frac{8\pi}{3}\boldsymbol{\nabla}\cdot\mathbf{P},\quad \boldsymbol{\nabla}\times\mathbb{E}=\frac{4\pi}{3}\boldsymbol{\nabla}\times\mathbf{P},\tag{11a}$$

$$\boldsymbol{\nabla}\cdot\mathbb{B}=-\frac{8\pi}{3}\boldsymbol{\nabla}\cdot\mathbf{M},\quad \boldsymbol{\nabla}\times\mathbb{B}=\frac{4\pi}{3}\boldsymbol{\nabla}\times\mathbf{M}.\tag{11b}$$

The solution of these equations is given by

$$\mathbb{E}=\int\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{P})-\mathbf{P}}{R^3}d^3x',\quad \mathbb{B}=\int\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{M})-\mathbf{M}}{R^3}d^3x'.\tag{12}$$

Equations (11) and (12) are invariant under the transformations

$$\mathbb{E}'=\mathbb{E}\cos\theta-\mathbb{B}\sin\theta,\quad \mathbf{P}'=\mathbf{P}\cos\theta-\mathbf{M}\sin\theta,\tag{13a}$$

$$\mathbb{B}'=\mathbb{E}\sin\theta+\mathbb{B}\cos\theta,\quad \mathbf{M}'=\mathbf{P}\sin\theta+\mathbf{M}\cos\theta,\tag{13b}$$

which follow from the duality transformations of the complex vectors $(\mathbb{E}+i\mathbb{B})$ and $(\mathbf{P}+i\mathbf{M})$. This means, for instance, that the solution in Eqs. (12) holds for primed quantities,

$$\mathbb{E}'=\int\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{P}')-\mathbf{P}'}{R^3}d^3x',$$

$$\mathbb{B}'=\int\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{M}')-\mathbf{M}'}{R^3}d^3x'.\tag{14}$$

Equations (11) and (12) can be transformed so that either polarization or magnetization is eliminated in the transformed equations. If θ is chosen to satisfy

$$\mathbf{P}=\mathbf{M}\tan\theta,\tag{15}$$

then $\mathbf{P}'=0$ and the transformed solution in Eqs. (14) reduces to

$$\mathbb{B}'=\int\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{M}')-\mathbf{M}'}{R^3}d^3x',\quad \mathbb{E}'=0.\tag{16}$$

Via this choice, the primed external electrostatic field disappears completely remaining only in the primed external magnetostatic field. It can be argued, however, that the insertion of $\mathbb{E}'=0$ and $\mathbf{P}'=0$ into Eqs. (13) implies $\mathbb{E}=\mathbb{B}'\sin\theta$ and $\mathbf{P}=\mathbf{M}'\sin\theta$ and thereby the electrostatic Maxwell's field $\mathbf{E}=\mathbb{E}-(4\pi/3)\mathbf{P}$ does not disappear in general:

$$\mathbf{E}=\mathbb{B}'\sin\theta-(4\pi/3)\,\mathbf{M}'\sin\theta.\tag{17}$$

This equation indicates that the field $\mathbf{E}$ can be expressed in terms of the field $\mathbb{B}'$ and the magnetization $\mathbf{M}'$ of the model presented here. This result deserves, however, some critical comments. Outside the sources field $\mathbf{E}$ coincides with field $\mathbb{E}$, which disappears *as an electrostatic* field ($\mathbb{E}'=0$) after performing the duality transformation, but it reappears *as a magnetostatic* field ($\mathbb{E}=\mathbb{B}'\sin\theta$) with regard to field $\mathbf{E}$. The final effect on field $\mathbf{E}$ is that it is seen as a magnetostatic field $\mathbf{E}=\mathbb{B}'\sin\theta$ when duality symmetry is used. In order to understand this last result consider an electric charge q_e at rest which is located inside field $\mathbf{E}=\mathbb{E}=\mathbb{B}'\sin\theta$. Traditional electrostatics states that q_e is urged by the force $\mathbf{F}=q_e\mathbf{E}=q_e\mathbb{E}$. However, the equivalent expression for the force

$$\mathbf{F}=q_e\mathbb{B}'\sin\theta\tag{18}$$

does not appear to have, at first sight, a simple interpretation, since any magnetic field (in this case $\mathbb{B}'$) should not act on a motionless electric charge. In order to answer this objection, consider Eqs. (2) and their associated duality transformations $(q_e'+iq_m')=e^{i\theta}(q_e+iq_m)$ from which one gets $q_m'=q_e\sin\theta+q_m\cos\theta$; this relation can also be derived from the second of Eqs. (13b) by assuming $\mathbf{P}=\mathbf{r}q_e\delta(\mathbf{x}-\mathbf{x}_0)$, $\mathbf{M}=\mathbf{r}q_m\delta(\mathbf{x}-\mathbf{x}_0)$ and $\mathbf{M}'=\mathbf{r}q_m'\delta(\mathbf{x}-\mathbf{x}_0)$, where $\mathbf{r}$ is a constant vector. The nonexistence of magnetic charges in the nonprimed quantities ($q_m=0$) implies $q_m'=q_e\sin\theta$, which is inserted into Eq. (18) to get $\mathbf{F}=q_m'\mathbb{B}'$. This expression for the force makes sense. Any magnetic field should act on a magnetic charge. Thus, the general result $q_e\mathbb{E}=q_m'\mathbb{B}'$ has been proved. By duality symmetry the force of an electrostatic field (due to a polarization) on a test electric charge is *equivalent* to the force of a magnetostatic field (due to a magnetization) on a test magnetic charge.

By a similar fashion, if θ is chosen to satisfy

$$\mathbf{P}=\mathbf{M}\bigl(-1/\tan\theta\bigr),\tag{19}$$

then $\mathbf{M}'=0$ and the transformed solution in Eqs. (14) becomes

$$\mathbb{E}'=\int\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{P}')-\mathbf{P}'}{R^3}d^3x',\quad \mathbb{B}'=0.\tag{20}$$

For this θ field $\mathbb{B}'$ disappears completely, and only field $\mathbb{E}'$ remains. The unification of the fields of the proposed model becomes clear: For θ satisfying Eq. (15) one gets the external magnetostatic field, and for θ satisfying Eq. (19) one gets the external electrostatic field. From Eqs. (13) and $\mathbb{B}'=0$ and $\mathbf{M}'=0$ one infers $\mathbb{B}=-\mathbb{E}'\sin\theta$ and $\mathbf{M}=-\mathbf{P}'\sin\theta$. Hence, the magnetostatic Maxwell's field $\mathbf{B}=\mathbb{B}+(8\pi/3)\mathbf{M}$ does not disappear in general,

$$\mathbf{B}=-\mathbb{E}'\sin\theta-(8\pi/3)\,\mathbf{P}'\sin\theta.\tag{21}$$

According to Eqs. (17) and (21), Maxwell's fields can be expressed by means of the fields and sources of the proposed $U(1)$ theory. From Eqs. (14), (15), and (19) one sees that the external electric and magnetic fields of material sources are exchanged, $\mathbb{B}'\leftrightarrow\mathbb{E}'$, if

$$\mathbf{M}'\leftrightarrow\mathbf{P}'\quad\text{and}\quad \alpha\leftrightarrow-1/\alpha,\tag{22}$$

where $\alpha=\tan\theta$. These exchanges show the power of duality: a question about magnetostatics may be elucidated by working with electrostatics, and vice versa. If there exists only an external magnetostatic field the coupling constant is $\alpha_M=\alpha$, and if there is only an external electrostatic field the coupling constant is $\alpha_E=-1/\alpha$. Equation (15) implies α

$=P/M$, where $P=|\mathbf{P}|$ and $M=|\mathbf{M}|$. This means that α measures how strongly or weakly the polarization is with respect to the magnetization, or equivalently, how strongly or weakly the magnetic dipole moment is with regard to the electric dipole moment, that is, $\alpha=d/\mu$ where $d=|\mathbf{d}|$ and $\mu=|\boldsymbol{\mu}|$. The constant α may be called the *dipole coupling constant*. For *weak* dipole coupling one means that the constant α is small. These results may be applied to a classical electron for which only a magnetostatic dipole field is observed. The magnetic moment of an electron is $\mu\approx 2\times 10^{-11}$ e cm. The current experimental value for a possible electric moment of the electron [16] is: $d=3\times 10^{-27}$ e cm. These values for d and μ yield $\alpha=1.5\times 10^{-16}$. Hence, the electric and magnetic dipole moments of the electron are *weakly* coupled. The exchanges in Eq. (22) illustrate, from a classical perspective, the admitted but not proven idea of duality in quantum theories.

The coupling of the polarization and magnetization sources in the static regime is one of the most remarkable results of Eqs. (11); such a coupling is not presented in Maxwell's theory. This coupling enables, for example, a dual description of a magnetostatic dipole field. Consider a particle number 1 (at the point $\mathbf{x}_0$) with a constant magnetic moment $\boldsymbol{\mu}$ and zero electric moment, $\mathbf{d}=0$. In this case $\mathbf{M}=\boldsymbol{\mu}\delta(\mathbf{x}-\mathbf{x}_0)$ and $\mathbf{P}=0$. With these specific sources, Eqs. (12) are integrated to get $\mathbb{B}=[3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}]/R^3$ and $\mathbb{E}=0$. Consider now particle number 2 (at the point $\mathbf{x}_0$) with an electric moment $\mathbf{d}'=-\boldsymbol{\mu}\sin\theta$ and a magnetic moment $\boldsymbol{\mu}'=\boldsymbol{\mu}\cos\theta$. In this case $\mathbf{P}'=-\boldsymbol{\mu}\sin\theta\delta(\mathbf{x}-\mathbf{x}_0)$ and $\mathbf{M}'=\boldsymbol{\mu}\cos\theta\delta(\mathbf{x}-\mathbf{x}_0)$. With these primed sources, Eqs. (14) are integrated to obtain $\mathbb{E}'=-\sin\theta\{[3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}]/R^3\}$ and $\mathbb{B}'=\cos\theta\{[3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}]/R^3\}$. These $\mathbb{E}'$ and $\mathbb{B}'$ and Eqs. (13) are used to yield a system of algebraic equations whose solution is $\mathbb{B}=[3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}]/R^3$ and $\mathbb{E}=0$. This means that particle number 1 is equivalent to particle number 2 in the sense that both particles yield the same effect. One can equally say that a magnetostatic dipole field is originated by a particle with a magnetic dipole moment $\boldsymbol{\mu}$; as one usually does, or that it is due to a particle with an electric dipole moment $\mathbf{d}'=-\boldsymbol{\mu}\sin\theta$ and a magnetic dipole moment $\boldsymbol{\mu}'=\boldsymbol{\mu}\cos\theta$. Similarly, an electrostatic dipole field may be interpreted as due to a particle possessing an electric moment $\mathbf{d}$, the usual interpretation, or as originated by particle having a magnetic moment $\boldsymbol{\mu}'=\mathbf{d}\sin\theta$ and an electric moment $\mathbf{d}'=\mathbf{d}\cos\theta$. A similar dual description of the fields of electric and magnetic charges has been pointed by Amaldi [17]. Equations (13), (15), and (19) can be directly established for time-dependent vectors. Hence, the extension of the above results to the dynamic regime of the theory is straightforward.

Finally, consider the polarization and magnetization vectors defined by

$$\mathbf{P}=\mathbf{n}R\rho/2$$

and

$$\mathbf{M}=(\mathbf{n}R\times\mathbf{J})/c\,, \tag{23}$$

where $\rho=\rho(\mathbf{x})$ and $\mathbf{J}=\mathbf{J}(\mathbf{x})$ are the usual charge and current densities that are assumed to be confined. Hence, the sources $\mathbf{P}$ and $\mathbf{M}$ in Eqs. (23) are confined. Insertion of these sources into Eqs. (12) yields the well-known Coulomb and Biot-Savart laws,

$$\mathbb{E}=\int\frac{\rho\mathbf{n}}{R^2}d^3x'$$

and

$$\mathbb{B}=\int\frac{\mathbf{J}\times\mathbf{n}}{R^2c}d^3x'. \tag{24}$$

The theory in Eqs. (11) contains, at least formally, the basic equations of electrostatics and magnetostatics. This result shows that the classical Maxwell's theory is not the unique picture in which Eqs. (24) appear in a natural way.

In conclusion, the exact symmetry of the external field lines of electric and magnetic dipoles has been used as a guide for constructing a classical model that describes and unifies the external electric and magnetic fields due to polarization and magnetization sources. It has been pointed out that this symmetry allows one to understand magnetostatics in terms of electrostatics and vice versa. In particular, the dual description of an external magnetostatic dipole field predicted by the model has been stressed. Some of the current ideas on duality in quantum theories are naturally realized in the classical model presented here, which offers a different perspective on the understanding of the electromagnetism of material sources and appears in the context of a recent revision of the background of Maxwell's theory [18].

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